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On the quasiparticle description of $c = 1$ CFTs

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Abstract

We show that the description of $c = 1$ conformal field theory in terms of quasiparticles satisfying fractional statistics can be obtained from the sine-Gordon model with a chemical potential A , in the limit where $A \gg M$. These quasiparticles are related to the excitations of the Calogero–Sutherland (CS) model. We provide a direct calculation of their two-particle S -matrix using Korepin’s method. We also reconsider the computation of the CS S -matrix in terms of particles with fractional charge.

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1. Introduction

Two-dimensional conformal field theories (CFTs) [1] and integrable quantum field theories (IQFTs) [2] are usually treated in a completely different fashion despite the fact that they are deeply related [3, 4]. A traditional starting point for the analysis of a CFT is the Hilbert space for the system in finite geometry, organized in terms of representations of chiral algebras for left and right moving excitations. On the other hand, IQFTs are conveniently described in terms of asymptotic particle states and the associated scattering data, pertaining to the system in infinite geometry. As a consequence of integrability the scattering is factorized, giving rise to a factorized scattering theory (FST). The two-particle S -matrix completely determines the on-shell dynamics as well as the off-shell properties such as correlation functions. In fact, once the exact particle spectrum and S -matrix are known, one can use their spectral representations together with the knowledge of the exact matrix elements, obtained using the form factor approach [5], to compute correlation functions of local fields. The factorized scattering approach is conceptually clear if the excitations in the field theory are massive, while, strictly speaking, scattering of massless relativistic particles is not well defined in two dimensions. Nevertheless, it has been found that, if the massless scattering is suitably defined, the FST approach can be very fruitful also for massless IQFTs [6–8].

In the context of applications to condensed matter systems in one spatial dimension, formulations of CFT in terms of FST become natural, both from a conceptual and from a computational point of view. Such formulations involve the identification of a suitable set of

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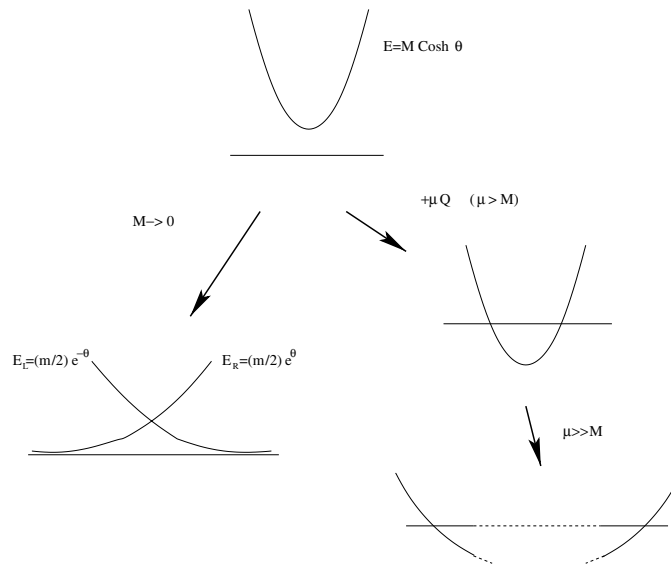


Figure 1. Schematic picture of the two different approaches to construct a quasiparticle representation of the $c = 1$ CFT from the sine-Gordon model.

(massless) CFT quasiparticles with factorized scattering, and the study of their two-particle S -matrix. For a particular class of $c = 1$ CFTs two interpretations in terms of FST have been considered. The first one is closely related to the usual approach to massive FSTs [6, 7]; the relevant S -matrix can be obtained as the massless limit, $M \rightarrow 0$, of the sine-Gordon (SG) S -matrix [9]. This particular description has been employed in the analysis of the edge-to-edge tunnelling in fractional quantum Hall (fqH) samples [10]. The second approach is intrinsic to the CFT [11, 12] and leads to a description in terms of a gas of quasiparticles that satisfy fractional exclusion statistics [13]. These quasiparticles have been identified with the excitations of the Calogero–Sutherland (CS) model in the continuum limit [16], and their S -matrix was inferred on the basis of a set of thermodynamic Bethe ansatz (TBA) equations [14, 15].

The aim of this paper is to explore the relationship between these two FST descriptions of $c = 1$ CFT. We will show that they emerge as different massless limits of the SG theory (see figure 1). In particular, the fractional statistics particles emerge as particle–hole excitations of the SG model in the presence of a chemical potential, $A \rightarrow \infty$ [18]. We provide a direct calculation of their S -matrix using Korepin’s method [17]. We also reconsider the calculation of the CS S -matrix [19] in terms of particles with fractional charge.

Our presentation is organized as follows. In the following section we review the standard construction of interacting CFT quasiparticles via a massless limit of the SG theory. In section 3 we introduce the fractional statistics quasiparticles in the CFT. Section 4 is devoted to fractional excitations in the CS model. In section 5 we show how the fractional statistics CFT quasiparticles can be obtained from the SG model in the presence of a chemical potential A , in the limit $A \rightarrow \infty$. We close with a brief discussion in section 6.

2. Massless limit of sine-Gordon: interacting quasiparticles

A quasiparticle description of $c = 1$ CFT can be obtained by taking the massless limit of a SG model [9]. In this section we recall a few basic facts about the SG theory and on the limit

$M \rightarrow 0$. The SG action is given by

$$S_{SG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 - 2\mu \cos(\beta\varphi) \right\} \tag{1}$$

where $\varphi(x)$ is a scalar field in 2D Euclidean spacetime $x = (x^0, x^1)$. This model possesses a $U(1)$ symmetry, generated by the charge

$$Q = \int_{-\infty}^{\infty} j^0 dx^1 = -\frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial x^1} dx^1 \tag{2}$$

where $j^\mu = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \varphi$ is the Noether current. From the perturbed CFT point of view, (1) can be considered as a Gaussian model

$$S_{Gauss} = \frac{1}{16\pi} \int d^2x (\partial_\nu \varphi)^2 \tag{3}$$

perturbed by the relevant operator $\exp(\pm i\beta\varphi)$ of scale dimension $\Delta_\beta = \beta^2$.

The SG model is integrable and has been studied in great detail over the last 25 years. The spectrum depends on the value of the coupling constant β^2 or, alternatively, on $\xi = \frac{\beta^2}{1-\beta^2}$. For $\beta^2 < 1$ the cosine term is relevant in the renormalization group sense and dynamically generates a spectral gap M in the excitation spectrum. In the repulsive regime $1 < \xi < \infty$ the spectrum contains only charged particles of charge $Q = \pm 1$, which are called solitons and antisolitons. The operators that generate these particles are [21, 22]

$$\mathcal{O}_{s,n}(x) = e^{i\frac{n}{4\beta} \tilde{\varphi}(x) + i\frac{s\beta}{n} \varphi(x)} \tag{4}$$

where the dual boson field, $\tilde{\varphi}$ is defined as

$$\tilde{\varphi}(x) = \int_{-\infty}^x \partial_y \varphi(x, y) dy. \tag{5}$$

These operators are in general non-local, and carry a spin s and a topological charge n . For $s = 1/2$, (4) corresponds to the bosonization formula for fermions of the massive thirring model [21], $\Psi(x) = \mathcal{O}_{\pm 1/2, 1}$, $\Psi^*(x) = \mathcal{O}_{\pm 1/2, -1}$.

In the attractive regime $0 < \xi < 1$ neutral soliton–antisoliton bound states, B_n , $n = 1, 2, \dots < 1/\xi$, called breathers, are formed and the spectrum becomes more complicated. One usually distinguishes soliton, antisolitons and breathers by some internal indices $\epsilon = s, \bar{s}$, B_n . The two-particle S -matrix, $S_{\epsilon_1, \epsilon_2}^{\epsilon'_1, \epsilon'_2}$, have been known for some time [2, 20]. We report here the soliton–soliton S -matrix, $S_{ss} = S_{ss}^{ss}$, which we need in the following:

$$S_{ss}(\theta) = -\exp[-i\delta_{ss}(\theta)] \quad \delta_{ss}(\theta) = \int_0^\infty \frac{\sin(\theta t/\pi) \sinh\left(\frac{1-\xi}{2}t\right)}{t \cosh\left(\frac{t}{2}\right) \sinh\left(\frac{\xi t}{2}\right)} dt. \tag{6}$$

Here the rapidity θ parametrizes the relativistic energy and momentum ($e(p) = \sqrt{p^2 + M^2}$)

$$p = M \sinh \theta \quad e = M \cosh \theta. \tag{7}$$

For large $|\theta|$, $\delta_{ss}(\theta)$ behaves like

$$\delta_{ss}(\theta) \simeq \pm\pi(\tilde{p} - 1) \quad \theta \rightarrow \pm\infty \tag{8}$$

with $\tilde{p} = 1/(2\beta^2)$.

An appropriate quasiparticle basis for the $c = 1$ CFT can be obtained taking the massless limit of SG particles [9, 10] described above. Formally this limit is constructed shifting the rapidities $\theta \rightarrow \theta \pm \theta_0/2$ and taking the limits $\theta_0 \rightarrow +\infty$, $M \rightarrow 0$ in such a way that $m = M \exp(\theta_0/2)$ remains finite. In this way one obtains the massless dispersion relations

$e = p = (m/2)e^\theta$, for right (R) movers and $e = -p = (m/2)e^{-\theta}$ for left (L) movers (where R and L movers are defined as $p > 0$ and $p < 0$ branches of the massless dispersion relation $e = \pm p$). Taking the same limit on the S -matrix one finds that the quasiparticle spectrum remains the same, i.e. it will have R and L solitons, antisolitons and breathers. While in the RR and LL sectors the S -matrix turns out to be the same as in the massive case, the RL (LR) scattering is trivial, i.e. equal to one. This is obviously related to the conformal symmetry. The presence of non-trivial RL scattering would signal a flow between critical points [6, 7]. The massless limit thus gives a FST of the CFT, with interacting quasiparticles with internal degrees of freedom and characterized by a non-diagonal S -matrix. The same result can be obtained by starting from the S -matrix axioms for unitarity, crossing and Yang–Baxter factorization directly for the massless particles [7].

At criticality the boson field, φ , can be decomposed into its holomorphic and antiholomorphic parts as

$$\varphi = \phi(z) + \bar{\phi}(\bar{z}) \quad \tilde{\varphi} = \phi(z) - \bar{\phi}(\bar{z}) \quad (9)$$

with $z = x + iy$ ($\bar{z} = x - iy$). The operators associated with massless solitons (antisolitons) take the form

$$\tilde{O}_{s,n} = e^{i(\frac{s\beta}{n} + \frac{n}{4\beta})\phi(z) + i(\frac{s\beta}{n} - \frac{n}{4\beta})\bar{\phi}(\bar{z})} \quad (10)$$

and the chiral components

$$\tilde{O}_{1/4\beta^2, \pm 1} = e^{\pm i\frac{1}{2\beta}\phi(z)} \quad \tilde{O}_{-1/4\beta^2, \pm 1} = e^{\mp i\frac{1}{2\beta}\bar{\phi}(\bar{z})} \quad (11)$$

correspond to the $U(1)$ conformal primary fields, J and \bar{J} , introduced in the following section. In this case the boson is compactified with a compactification radius $R = 1/(2\beta)$.

3. CFT quasiparticles with fractional statistics

In this section we review the construction of a quasiparticle basis for (chiral) $c = 1$ CFT, which is very different from the direct $M \rightarrow 0$ limit of the SG particle basis. This basis is a special example of a ‘CFT quasiparticle basis’ [11]. The idea behind the construction of the CFT quasiparticles is to first classify the chiral primary fields, and then associate quasiparticles with specific chiral primaries (usually those with the smallest conformal dimensions). To construct an actual basis of the chiral Hilbert space one needs to impose specific rules on the modes (momenta) of the quasiparticles that participate in a multi-particle state; these rules then lead to a statement about a form of exclusion statistics satisfied by the quasiparticles.

The prototype of a CFT quasiparticle basis is the so-called spinon basis of the $c = 1$ $SU(2)$ and $SU(N)$ invariant CFTs [23]; the more general construction was first outlined in [11]. The particular example we deal with here is the $c = 1$ CFTs at compactification radius $R^2 = p/2$. In these theories, the chiral primary of smallest dimension carries a $U(1)$ charge $\pm\frac{1}{p}$, and one may contemplate quasiparticle bases built out of these fractionally charged quanta. This is particularly natural from the point of view of the fqH systems at filling fraction $\nu = \frac{1}{p}$, which have this particular CFT as effective edge theories [12].

The most natural way to build a CFT quasiparticle basis at $c = 1$, $R^2 = p/2$, involves quasiholes Ψ_{qh} , of charge $+\frac{1}{p}$, and particles J , of charge -1 . They are described by the conformal primary fields

$$J(z) = e^{-i\sqrt{\frac{p}{2}}\phi(z)} = \sum_t J_{-t} z^{t-p/2} \quad \Psi_{qh}(z) = e^{i\frac{1}{\sqrt{2p}}\phi(z)} = \sum_s \phi_{-s} z^{s-1/2p} \quad (12)$$

(note that the conventions used here are different from those in [12]). The J operators correspond to the massless limit of soliton creation operators (11) with the identification $p = \tilde{p} = 1/2\beta^2$.

The independent multi-particle states that generate the chiral Hilbert space were identified to be

$$|m_M, \dots, m_1; n_N \dots n_1\rangle^Q \equiv J_{-(2M-1)p/2+Q-m_M} \dots J_{-p/2+Q-m_1} \phi_{-(2N-1)/2p-Q/p-n_N} \dots \phi_{-1/2p-Q/p-n_1} |Q\rangle \tag{13}$$

with

$$m_M \geq \dots \geq m_1 \geq 0 \quad n_N \geq \dots \geq n_1 \geq 0 \quad (n_1 > 0 \text{ if } Q < 0) \tag{14}$$

where $|Q\rangle$ ($Q = -(p-1), \dots, -1, 0$) is the lowest energy state of charge Q/p .

Using this basis, one can analyse the partition sum, and thereby the thermodynamic properties, directly in terms of the quasiparticles J and ϕ . One then finds that the thermodynamic equations take the form of the so-called IOW equations [14] that describe the thermodynamics of a gas of particles satisfying fractional exclusion statistics. For the general case with statistics matrix $\mathbf{g} = (g_{ij})$, the excitation energies, ε_i , and distribution functions, \bar{n}_i , are determined by the following equations:

$$\left(\frac{\lambda_i - 1}{\lambda_i}\right) \prod_j \lambda_j^{g_{ij}} = e^{\beta(\mu_i - \varepsilon_i)} \equiv z_i \quad \bar{n}_i(\varepsilon_i) = z_i \frac{\partial}{\partial z_i} \log \prod_j \lambda_j \tag{15}$$

with $\lambda_i = (1 + e^{-\varepsilon_i})$ being the one-particle grand canonical partition functions. For the case of $c = 1$ one finds the following statistical parameters: $g_{\phi\phi} = 1/p$, $g_{JJ} = p$ and $g_{J\phi} = g_{\phi J} = 0$.

Summarizing, we see that the $c = 1$, $R^2 = p/2$ chiral CFT is described by quasiparticles with charge/statistics parameters ($Q = \frac{1}{p}$, $g = \frac{1}{p}$) and ($Q = -1$, $g = p$) and with no mutual statistics between the two.

It is well known [14, 15] that the IOW equations for statistics matrix g_{ij} agree with the TBA equations for a ‘purely statistical’ two-particle S -matrix given by

$$S_{ij}(\theta) = -\exp[\pi i(\delta_{ij} - g_{ij}) \text{sign}(\theta)] \tag{16}$$

where $\text{sign}(x) = |x|/x$ is the sign function ($\text{sign}(0) = 0$). Combining this with the above, we tentatively identify the above CFT quasiparticle basis with a particle basis in the sense of FST. In the remainder of this paper, we substantiate this claim by establishing the relation between the CFT and this particular FST from two alternative points of view, which are the Calogero–Sutherland model and an alternative ($A \rightarrow \infty$) massless limit of the SG model.

4. S-matrix for fractional excitations in the CS model

The CFT quasiparticles described in the previous section are particularly natural if the CFT is viewed as the continuum limit of a so-called Calogero–Sutherland model for particles with inverse-square interaction, as the CFT quasiparticles can be identified with the fundamental excitations of the CS model [16]. In this section we evaluate the S -matrix of the CS quasiparticles using a method developed by Korepin [17]. Here and in the following we will focus our attention only on the computation of the scattering amplitudes between right movers. The computation of the L–L amplitude is equivalent and the result has to be the same for symmetry reasons. A related calculation was done in [19]; here we use a different scheme based on excitations with fractional charge.

The CS model [24–26] describes fermionic particles on a line whose interaction, for N particles, is given by the following Hamiltonian:

$$H = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j < k} \frac{2p(p-1)}{(x_k - x_j)^2}. \tag{17}$$

In the low energy sector it is described by $c = 1$ CFT with compactification radius that, using the conventions of section 3, is $R^2 = p/2$ [16]. The model was solved with the Bethe ansatz (BA) in [25]. Imposing periodic boundary conditions on the wavefunction, one obtains the following quantization (BA) equations [25]:

$$L\lambda_i = 2\pi I_i + \sum_{j \neq i}^N \theta(\lambda_i - \lambda_j) \quad i = 1, \dots, N \quad (18)$$

where

$$\theta(\lambda) = (p - 1)\pi \operatorname{sign}(\lambda). \quad (19)$$

The set of integers or half-odd integers I_i is in one-to-one correspondence with a set of spectral parameters that specify an eigenstate of the Hamiltonian. The total energy is parametrized as $E = \sum_i \lambda_i^2$. In the presence of chemical potential, μ , the ground state is obtained by filling the Fermi sea with rapidities $|\lambda| < \sqrt{\mu} \equiv \lambda_F$, at distances $\lambda_i - \lambda_j > \frac{2\pi(p-1)}{L}$ (this is the first signal of a generalized exclusion statistics). The Fermi momentum λ_F can be written in terms of the total number of λ in the condensate, N_{GS} , as $\lambda_F = \pi p(N_{GS} - 1)/L$. The integers (half-odd integers) I_i vary in the interval

$$|I_j| < I_{\max}^{GS} \quad (20)$$

with

$$I_{\max}^{GS} = \frac{1}{4\pi} \left| \lambda_F L - \sum_k^{N_{GS}} \theta(+\infty) \right| = \frac{1}{2}(N_{GS} - 1). \quad (21)$$

There are different types of possible excitations.

I: Quasihole excitations. The lowest energy hole-type excitation corresponds to occupation numbers $N = N_{GS} - 1$. If we keep the Fermi momentum, λ_F , fixed to order one, the range of the integers I_i changes according to (21)

$$I_{\max} = I_{\max}^{GS} + \frac{p-1}{2}. \quad (22)$$

Then one has $p - 1$ additional vacancies and one less λ , i.e. the excitations are characterized by p parameters. We can say that this excitation corresponds to creating p quasiholes, $\lambda_h^1, \dots, \lambda_h^p$, in the ground state, each having charge $1/p$. We use the term quasihole to distinguish these excitations from the charge-1 hole excitations that we will discuss in the following. The p -quasihole excitations are similar to the $S = 1$ 2-spinon excitations in the $S = 1/2$ Heisenberg [27] or Haldane–Shastry models [19].

The BA equations relative to this type of excitation are the following:

$$L\tilde{\lambda}_i = 2\pi \tilde{I}_i + \sum_{j \neq i}^{N_{GS}+(p-1)} \theta(\tilde{\lambda}_i - \tilde{\lambda}_j) - \sum_{a=1}^p \theta(\tilde{\lambda}_i - \lambda_a^h) \quad (23)$$

where $\tilde{\lambda}$ indicates that the spectral parameters have slightly changed with respect to the ground state distribution ($\tilde{\lambda}_i - \lambda_i \sim O(1/L)$). We follow the conventions of [28], take $\tilde{I}_i - I_i = (p - 1)/2$ for $i = 1, \dots, N_{GS}$, and place the additional vacancies, $\lambda_1^{\min}, \dots, \lambda_{p-1}^{\min}$, close to the left Fermi point. In the thermodynamic limit $\lambda_i^{\min} \rightarrow -\infty$. As usual, we characterize these excitations via the shift function [17]

$$F_{hh}(\lambda_i) = \frac{\tilde{\lambda}_i - \lambda_i}{\lambda_{i+1} - \lambda_i}. \quad (24)$$

The equation defining $F_{hh}(\lambda)$ can be obtained subtracting (23) from the ground state distribution and using the fact that $\tilde{\lambda}_i - \lambda_i \sim O(1/L)$,

$$F_{hh}(\lambda_i) + \frac{1}{2\pi} \sum_{j \neq i}^{N_{GS}} K(\lambda_i - \lambda_j) F_{hh}(\lambda_j) = \frac{p-1}{2} + \frac{1}{2\pi} \sum_{k=1}^{p-1} \theta(\lambda_i - \lambda_k^{\min}) - \frac{1}{2\pi} \sum_{a=1}^p \theta(\lambda_i - \lambda_a^h) \tag{25}$$

where $K(\lambda) = \theta'(\lambda) = 2\pi(p-1)\delta(\lambda)$. In the thermodynamic limit it becomes

$$F_{hh}(\lambda) + \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\mu K(\lambda - \mu) F_{hh}(\mu) = \frac{p-1}{2} + \frac{1}{2\pi} (p-1)\theta(+\infty) - \frac{1}{2\pi} \sum_{a=1}^p \theta(\lambda - \lambda_a^h). \tag{26}$$

Using the explicit form of $K(\lambda)$ and $\theta(\lambda)$ we find

$$F_{hh}(\lambda_h^1) = \frac{p-1}{2p} \quad \lambda_h^1 > \lambda_h^b (b = 2, \dots, p). \tag{27}$$

II: Particle excitations. Particle excitations correspond to $N = N_{GS} + 1$, where the additional particle has rapidity $|\lambda_p| > \lambda_F$. The BA equations have the form

$$L\tilde{\lambda}_i = 2\pi\tilde{I}_i + \sum_{j \neq i}^{N_{GS}} \theta(\tilde{\lambda}_i - \tilde{\lambda}_j) + \theta(\tilde{\lambda}_i - \lambda_p) \tag{28}$$

$$L\lambda_p = 2\pi\tilde{I}_{N_{GS}+1} + \sum_j^{N_{GS}} \theta(\lambda_p - \tilde{\lambda}_j). \tag{29}$$

In the thermodynamic limit the integral equations for F have the form

$$F_{pp}(\lambda) + \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\mu K(\lambda - \mu) F_{pp}(\mu) = \theta(\lambda - \lambda_p) \tag{30}$$

with the explicit solution

$$F_{pp}(\lambda_p) = \frac{(p-1)}{2}. \tag{31}$$

From these two types of excitations it is possible to construct neutral excitations corresponding to one particle and p quasiholes. The shift function associated with these types of excitations is defined by the equation

$$\begin{aligned} F_{ph}(\lambda) + \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\mu K(\lambda - \mu) F_{ph}(\mu) \\ = \frac{p-1}{2} + \frac{1}{2\pi} \theta(\lambda - \lambda_p) + \frac{1}{2\pi} (p-1)\theta(+\infty) - \frac{1}{2\pi} \sum_{a=1}^p \theta(\lambda - \lambda_a^h) \end{aligned} \tag{32}$$

from which

$$F_{ph}(\lambda_p) = 0 \quad \text{mod } 2\pi. \tag{33}$$

We choose these types of excitations to describe the Hilbert space and construct the FST. Below we explicitly calculate their S -matrix.

In [19] the physical excitations were constructed using a different scheme that corresponds to allowing the Fermi momentum in (21) to vary of order $O(1/L)$ when one creates a hole

excitation. Within this scheme, changing the occupation numbers by one ($N_{GS} \rightarrow N_{GS} - 1$) produces a shift of I_{\max} by one. In this way all the excitations have integer charge. On the level of the BA the two descriptions seem to be complementary. For reasons that will become clear later we prefer to use as basic excitations I and II, i.e. particles with integer charge but quasiholes with a fractional charge. This approach is closer to the philosophy of [29, 30].

S-matrix for CS model. We now apply Korepin's method [17] to compute the S -matrix relative to the excitations I and II described above. The S -matrix is just a phase

$$S_{ab} = \exp(-i\delta_{ab}) \quad (34)$$

where the two-particle scattering phase is equal to the phase, φ_{ab} , obtained by moving the particle a through the system in the presence of particle b minus the phase shift, φ_a , obtained through the same process but in the absence of particle b

$$\delta_{ab} = \varphi_{ab} - \varphi_a. \quad (35)$$

Both phases can be obtained using the BA equations for the ground state and excitations. In general, it turns out [17] that δ_{ab} is related to the shift function via

$$\delta_{ab}(\lambda_i, \lambda_j) |_{\lambda_i > \lambda_j} = 2\pi F_{ab}(\lambda_i). \quad (36)$$

Unfortunately, this approach cannot be applied directly to the quasihole excitations introduced in the previous section. In fact, within the scheme we consider here, it is not possible to create one- and two-quasihole excitations and thus it is not possible to evaluate directly the one- and two-particle phase shift. Nevertheless it is possible to evaluate the total phase shift associated with an excitation I (consisting of p quasiholes) as a whole. This corresponds to the phase shift acquired by the fastest particle going across the other $p - 1$ quasiholes. Using factorization, together with the fact that the resulting S -matrix is momentum independent, one can see that the quasihole–quasihole phase shift, φ_{hh} , is given by

$$\varphi_{hh}(\lambda_h^1, \lambda_h^2) |_{\lambda_h^1 > \lambda_h^2} = 2\pi F_{hh}(\lambda_h^1) / (p - 1) = \pi / p \quad (\lambda_h^1 > \lambda_h^b, b = 2, \dots, p). \quad (37)$$

In order to compute δ_{hh} we should now subtract the one-particle phase shift, φ_h . Having no direct access to φ_h , we subtract a reference phase, $\delta_0(p)$,

$$\delta_{hh}(\lambda_h^1, \lambda_h^2) |_{\lambda_h^1 > \lambda_h^2} = 2\pi F_{hh}(\lambda_h^1) / (p - 1) - \delta_0(p) = \pi / p - \delta_0(p) \quad (\lambda_h^1 > \lambda_h^b, b = 2, \dots, p). \quad (38)$$

On the basis of the observation that individual quasiholes are local with respect to the CS ground state, we anticipate that this reference phase will be an integer multiple of π . Below we shall fix its value by an independent argument.

Following a similar reasoning we find the particle–quasihole phase shift,

$$\delta_{ph}(\lambda_p, \lambda_h^1) = 2\pi F_{ph}(\lambda_p) / p = 0 \pmod{2\pi}. \quad (39)$$

The particle–particle S -matrix can be computed in a more standard way [19] since one- and two-particle excitations can be constructed explicitly. As a consequence, the two-particle S -matrix will be completely determined. One finds

$$\delta_{pp}(\lambda_p^1, \lambda_p^2) |_{\lambda_p^1 > \lambda_p^2} = 2\pi F_{pp}(\lambda_p^1) = (p - 1)\pi. \quad (40)$$

The identification of (38), (39) and (40) as phase shifts clearly requires $\lambda_1 > \lambda_2$. Analytic continuation of these results to the sector $\lambda_1 < \lambda_2$ can be done using the unitarity condition of the S -matrix: $S_{ab}(\lambda)S_{ab}(-\lambda) = 1$. From which it follows:

$$S_{ab}(\lambda) = -\exp[-i\delta_{ab} \text{sign}(\lambda)] \quad a, b = p, h \quad (41)$$

where we have chosen a ‘fermionic’ normalization, $S_{ab}(0) = -1$ [31]. In order to fix the reference phase, $\delta_0(p)$, in (38) we use the known duality of the CS Hamiltonian that, under $p \rightarrow 1/p$, maps particles into holes and vice-versa [32]. This implies $\delta_0(p) = \pi$, in agreement with our expectation. Our full result for the two-particle S -matrix is in agreement with what was found using different methods [12] as reported in section 3.

We note that the quasihole–quasihole scattering phase has a dependence typical of particles with fractional statistics, this is the reason for the choice of the scheme above. Our results are consistent, we have particles with fractional charge that satisfy fractional exclusion statistics.

5. Fractional CFT quasiparticles from sine-Gordon

Let us now consider the SG model in the presence of a chemical potential coupled to the conserved charge. The Hamiltonian is shifted as

$$\mathcal{H}(A) = \mathcal{H}_{SG} - AQ. \tag{42}$$

The potential A works as infrared cut-off at scales, i.e. the excitations will have an energy of order A . Therefore, since the cosine term in (1) is bounded, for $A \gg \mu^{(p+1)/2}$ the theory is driven to the UV fixed point [18], where it is described by the $c = 1$ CFT (3). In the presence of the chemical potential every soliton (antisoliton) acquires an additional energy $A(-A)$, while the breathers spectrum is not affected. For $A > M$ the ground state is a soliton condensate and the theory has massless excitations of the particle–hole type across the Fermi sea. The other excitations, associated with antisolitons and breathers, have a gap, so we do not need to consider them for what follows. Using Korepin’s method [17], we shall construct the S -matrix for excitations over the soliton condensate in the limit $A \rightarrow \infty$, and show that they are free particles satisfying fractional statistics. These quasiparticles then provide a FST description of the $c = 1$ fixed point. We will find that the S -matrix is the same as that for the CS model constructed in the previous section.

We can repeat the analysis of the excitations and S -matrix done for the CS model in section 4 for the SG model in the presence of the chemical potential $A > M$ (42). Putting N solitons on the space line of length L and imposing periodic boundary conditions one obtains the following quantization equations [31]:

$$Lp(\theta_i) = 2\pi I_i + \sum_{j \neq i}^N \delta_{ss}(\theta_i - \theta_j) \quad i = 1, \dots, N \tag{43}$$

where $\delta_{ss}(\theta)$ and $p(\theta)$ are defined in (6) and (7), respectively. The presence of the chemical potential induces also in this case a particle condensate and the ground state is obtained filling the rapidities symmetrically around zero. In the thermodynamic limit the ground state energy in the presence of the chemical potential is given by

$$\mathcal{E}(A) - \mathcal{E}(0) = \frac{M}{2\pi} \int_{-B}^B d\theta \cosh \theta \epsilon(\theta) \tag{44}$$

where $\epsilon(\theta)$ is a non-positive function defined by the following equation:

$$\epsilon(\theta) + \int_{-B}^B d\theta' K_{ss}(\theta - \theta') \epsilon(\theta') = M \cosh \theta - A \quad \epsilon(\pm B) = 0 \tag{45}$$

with

$$K_{ss}(\theta) = \frac{1}{2\pi} \frac{d\delta_{ss}(\theta)}{d\theta}. \tag{46}$$

In momentum space the kernel $\tilde{K}_{ss}(\theta) = \delta(\theta) + K_{ss}(\theta)$ has a quite simple form

$$\tilde{K}_{ss}(\omega) = \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} \tilde{K}_{ss}(\theta) = \frac{\sinh \frac{\pi(1+\xi)\omega}{2}}{2 \cosh \frac{\pi\omega}{2} \sinh \frac{\pi\xi\omega}{2}} \tag{47}$$

and can be factorized as

$$\tilde{K}_{ss}(\omega) = \frac{1}{K_+(\omega)K_-(\omega)} \tag{48}$$

with

$$K_+(\omega) = K_-(-\omega) = \sqrt{\frac{2\pi(\xi+1)}{\xi}} e^{i\omega\Delta} \frac{\Gamma(i\frac{\xi+1}{2}\omega)}{\Gamma(i\frac{\xi\omega}{2})\Gamma(\frac{1}{2} + i\frac{\omega}{2})} \tag{49}$$

analytic in the upper (K_+) and lower (K_-) half plane. In (49) $\Delta = \frac{\xi}{2} \log \xi - \frac{\xi+1}{2} \log(\xi+1)$ so that $K_+(\omega) = 1 + O(1/\omega)$. The limit $A \rightarrow \infty$ corresponds to $B \rightarrow \infty$ and equations (44), (45) become [18]

$$\mathcal{E}(A) - \mathcal{E}(0) = \frac{M}{4\pi} \int_{-B}^B d\theta e^{\theta} \epsilon(\theta) \tag{50}$$

$$\int_{-\infty}^B d\theta' \tilde{K}_{ss}(\theta - \theta') \epsilon(\theta') = \frac{M}{2} e^{\theta} - A \tag{51}$$

clearly describing a massless system.

The excitations can be studied using (43). They turn out to be similar to those discussed in section 4 for the CS model. Also in this case we find quasihole excitations with fractional charge, the reason being that $\delta_{ss}(+\infty) \neq 1$, and removing a θ produces a shift of I_{\max} according to (21). From equation (8) we see that this shift produces $\tilde{p} - 1$ additional vacancies. Then again an excitation will be characterized by \tilde{p} parameters and can be interpreted as consisting of \tilde{p} quasiholes of charge $1/\tilde{p}$ in the ground state. Below we show that, in the limit $A \rightarrow \infty$, the S -matrix relative to these excitations is again given by (38), (39) and (40), where \tilde{p} replaces the CS parameter p . With this identification the CS model and the UV limit of the SG model give rise to the same quasiparticle S -matrix. For any finite A there are corrections to the CS S -matrix that depend on the rapidities.

Let us first consider quasihole excitations. They will be characterized by the following BA equations:

$$Lp(\tilde{\theta}_i) = 2\pi \tilde{I}_i + \sum_{j \neq i}^{N_{GS}+(\tilde{p}-1)} \delta_{ss}(\tilde{\theta}_i - \tilde{\theta}_j) - \sum_{a=1}^{\tilde{p}} \delta_{ss}(\tilde{\theta}_i - \theta_h^a) \tag{52}$$

and can be studied again introducing the shift function, $\tilde{F}_{hh}(\theta_i) = (\tilde{\theta}_i - \theta_i)/(\theta_{i+1} - \theta_i)$, satisfying the following equation (in the thermodynamic limit):

$$\begin{aligned} \tilde{F}_{hh}(\theta) + \frac{1}{2\pi} \int_{-B}^B d\theta' K_{ss}(\theta - \theta') \tilde{F}_{hh}(\theta') \\ = \frac{\tilde{p} - 1}{2} + \frac{1}{2\pi} (\tilde{p} - 1) \delta_{ss}(+\infty) - \frac{1}{2\pi} \sum_{a=1}^{\tilde{p}} \delta_{ss}(\theta - \theta_h^a). \end{aligned} \tag{53}$$

In the limit $B \rightarrow \infty$, equation (53) can be solved with the Wiener–Hopf (WH) method (see, for instance, appendix B of [18]). The procedure is quite standard and we report here only the

essential steps. Shifting $\theta_h^1 \rightarrow \theta_h^1 + B$ and introducing $f_{hh}(\theta_h^1) = F_{hh}(\theta_h^1 + B)$ one can rewrite (53) as

$$f_{hh}(\theta_h^1) + \frac{1}{2\pi} \int_{-2B}^0 d\theta' K_{ss}(\theta_h^1 - \theta') f_{hh}(\theta') = \frac{\tilde{p} - 1}{2} + \frac{1}{2\pi} (\tilde{p} - 1) \delta_{ss}(+\infty) - \frac{1}{2\pi} \sum_{a=1}^{\tilde{p}-1} \delta_{ss}(\theta_h^1 - \theta_a^h + B). \tag{54}$$

As a consequence of the shift, quasihole excitations now correspond to $\theta_1^h < 0$. For $B \rightarrow \infty$ equation (54) takes the form

$$f_{hh}(\theta_h^1) + \frac{1}{2\pi} \int_{-\infty}^0 d\theta' K_{ss}(\theta_h^1 - \theta') f_{hh}(\theta') = g_\infty(\tilde{p}) \tag{55}$$

where on the RHS we have approximated $\delta_{ss}(\theta)$ with its asymptotics

$$g_\infty(\tilde{p}) = \frac{\tilde{p} - 1}{2} + \frac{1}{2\pi} (\tilde{p} - 1) \delta_{ss}(+\infty) - \frac{1}{2\pi} \sum_{a=1}^{\tilde{p}-1} \delta_{ss}^\infty = \frac{\tilde{p} - 1}{2}. \tag{56}$$

This is the same driving term as for the CS model (26). We can solve this equation for any θ although we only need it for $\theta < 0$. We rewrite $f(\theta)$ as $f(\theta) = f^+(\theta) + f^-(\theta)$, where $f^+(\theta) = f(\theta)$ for $\theta > 0$ and zero otherwise, and $f^-(\theta) = f(\theta)$ for $\theta < 0$. Fourier transforming, equation (55) becomes

$$f^+(\omega) + \tilde{K}_{ss}(\omega) f^-(\omega) = g_\infty(\tilde{p}) \delta(\omega) \tag{57}$$

and can be solved with the WH method, to give

$$f^+(\omega) = g_\infty(\tilde{p}) \frac{K_+(0)}{K_+(\omega)} \frac{1}{\omega + i0} \quad f^-(\omega) = g_\infty(\tilde{p}) K_+(0) \frac{K_-(\omega)}{\omega - i0}. \tag{58}$$

We can now obtain $F_{hh}(\theta_h^1)$ by Fourier transforming equation (58)

$$F_{hh}(\theta_h^1) = \int d\omega e^{i\omega(\theta_h^1 + B)} f^-(\omega) \sim g_\infty(\tilde{p}) K_+(0) K_-(0) = \frac{g_\infty(\tilde{p})}{\tilde{K}_{ss}(0)} = \frac{\tilde{p} - 1}{2\tilde{p}} \tag{59}$$

where we have omitted terms of order $O(\exp(-B))$. This is the same as for the CS model with the identification $\tilde{p} = p$, as previously anticipated. Following the same steps as in section 4, we obtain also $\tilde{\delta}_{ph} = 0$ and $\tilde{\delta}_{pp} = (\tilde{p} - 1)\pi$. Although it is not possible to show it on the basis of the BA, it is quite natural to argue that these excitations are generated by the operators (12).

6. Discussion

In this paper we show how fractional statistics quasiparticles in specific $c = 1$ CFTs can be obtained from an associated sine-Gordon model. Introducing a chemical potential, A , and driving the system to the UV fixed point by taking $A \rightarrow \infty$, we constructed massless excitations with fractional charge and computed their S -matrix. These excitations correspond to the excitations of the Calogero–Sutherland model associated with the same CFT. Their S -matrix is momentum independent, giving rise to the notion of a free gas of particles with generalized statistics. This formulation of the $c = 1$ CFT can be contrasted with the formulation obtained via a massless limit, $M \rightarrow 0$, of the same sine-Gordon theory. Our result thus sheds some light on the relation between different factorized scattering theories associated with a $c = 1$ CFT. It will be worthwhile to explore similar relations for FST formulations of more general (rational) CFTs.

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